Journal of HyoJeong Academia

Article

Effective parameter evaluation of cylindrically propagating elastic waves

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Abstract: This study proposes a methodology to interpret cylindrically propagating elastic waves as analogous to waves propagating in Cartesian coordinates. By employing coordinate transformations and leveraging the symmetry of wave propagation, the approach simplifies the complexities inherent in cylindrical coordinate systems. This framework enables the application of existing Cartesian-based analytical techniques to cylindrical wave problems, providing enhanced computational efficiency. The proposed method holds potential for various engineering applications, including elastic wave-based nondestructive testing and structural optimization, particularly in complex media and structures.

Keywords: Elastic metamaterial, Wave propagation, Transfer matrix

1. Introduction

Understanding the elastic effective properties of materials is crucial in a wide range of engineering applications, from aerospace structures to biomedical devices [1,2]. These properties provide insights into the material's ability to withstand mechanical forces, propagate stress waves, and perform under dynamic conditions. In recent years, the study of wave propagation in complex materials has garnered significant attention, as it offers a deeper understanding of the interplay between material structure and mechanical behavior.

Among the various methodologies available for studying wave propagation, the transfer matrix method (TMM) has emerged as a powerful analytical and numerical tool [3,4]. Its ability to efficiently model wave behavior in layered media and periodic structures makes it particularly well-suited for exploring cylindrically propagating waves. Cylindrical wave propagation, which is characterized by its radial symmetry and dependence on azimuthal and radial coordinates, is highly relevant in scenarios involving cylindrical geometries, such as pipelines, acoustic sensors, and composite cylindrical shells.

Despite the extensive research on TMM and its applications to planar wave propagation, relatively few studies have focused on its application to cylindrical wave propagation, especially in the context of determining elastic effective properties. This gap in the literature presents a significant opportunity to extend the utility of TMM to cylindrical geometries, enabling accurate and efficient modeling of wave mechanics in these systems.

This study aims to develop a comprehensive framework for determining the elastic effective properties of materials subjected to cylindrically propagating waves using the transfer matrix method. By leveraging the inherent advantages of TMM, such as its modularity and ability to handle multi-layered systems, this approach provides a robust means of analyzing wave behavior in complex cylindrical configurations. Additionally, the findings of this study will have implications for the design and optimization of materials and structures with tailored wave propagation characteristics.

Citation: Chung Il Park. Effective parameter evaluation of cylindrically propagatiang elastic waves. *JHJA* 3(2), **2025**.

Received: August 25, 2025
Accepted: October 19, 2025
Published: October 24, 2025

Publisher's Note: JHJA stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

In the following sections, one first presents a detailed overview of the theoretical foundation of the transfer matrix method as applied to cylindrical wave propagation. It then describes the derivation of the elastic effective properties and validates the proposed methodology. Finally, one discusses the broader implications of this research and suggests potential directions for future studies in this domain.

2. Materials and Methods

The wave propagation of a longitudinal wave on isotropic elastic medium in the cylindrical coordinate can be described by using scalar potential function φ:

$$\varphi(\omega; r,t) = AH_\nu^{(1)}(k_r r)e^{-i\omega t} + BH_\nu^{(2)}(k_r r)e^{-i\omega t} \tag{1}$$

where ω denotes angular frequency and r and t respectively represent radial coordinate and time. A and B are wave amplitude for positive *r*-going and negative *r*-going wave respectively. $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are Hankel function of the first and second kind respectively with order number ν. k_r represents wavenumber along the *r*-direction. Here, for simplicity, the *r*-going wave is considered only. The wave number along the θ and z directions, k_θ and k_z, are set to be zero which otherwise can be written as $\nu = k_\theta = 0$ i is the imaginary unit. This scalar longitudinal wave potential provides displacement and strain field as

$$u_r = \frac{\partial\varphi}{\partial r} = Ak_r\left(\frac{H_{\nu-1}^{(1)}(k_r r) - H_{\nu+1}^{(1)}(k_r r)}{2}\right) + Bk_r\left(\frac{H_{\nu-1}^{(2)}(k_r r) - H_{\nu+1}^{(2)}(k_r r)}{2}\right) \tag{2a}$$

$$u_\theta = u_z = 0 \tag{2b}$$

$$\varepsilon_{rr} = \frac{\partial^2\varphi}{\partial r^2} = Ak_r^2\left(\frac{H_{\nu-2}^{(1)}(k_r r) - 2H_\nu^{(1)}(k_r r) + H_{\nu+2}^{(1)}(k_r r)}{4}\right) + Bk^2\left(\frac{H_{\nu-2}^{(2)}(k_r r) - 2H_\nu^{(2)}(k_r r) + H_{\nu+2}^{(2)}(k_r r)}{4}\right) \tag{3a}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} = A\frac{k_r}{r}\left(\frac{H_{\nu-1}^{(1)}(k_r r) - H_{\nu+1}^{(1)}(k_r r)}{2}\right) + B\frac{k_r}{r}\left(\frac{H_{\nu-1}^{(2)}(k_r r) - H_{\nu+1}^{(2)}(k_r r)}{2}\right) \tag{3b}$$

$$\varepsilon_{zz} = \varepsilon_{r\theta} = \varepsilon_{\theta z} = \varepsilon_{zr} = 0 \tag{3c}$$

Here the time harmonic term $e^{-i\omega t}$ is omitted for simplicity without loss of generality. The propagating medium is assumed to be isotropic satisfying $\sigma_{ij} = \lambda e\delta_{ij} + 2\mu\varepsilon_{ij}\,(i, j = r, \theta, z)$, which in turn provides velocity and stress field as

$$v_r = \frac{\partial u_r}{\partial tr} = -i\omega Ak_r\left(\frac{H_{\nu-1}^{(1)}(k_r r) - H_{\nu+1}^{(1)}(k_r r)}{2}\right) - i\omega Bk_r\left(\frac{H_{\nu-1}^{(2)}(k_r r) - H_{\nu+1}^{(2)}(k_r r)}{2}\right) \tag{4}$$

$$\begin{aligned}\sigma_{rr} = (\lambda + 2\mu)\varepsilon_{rr} + \lambda\varepsilon_{\theta\theta} = &\left\{(\lambda + 2\mu)k_r^2\left(\frac{H_{\nu-2}^{(1)}(k_r r) - 2H_\nu^{(1)}(k_r r) + H_{\nu+2}^{(1)}(k_r r)}{4}\right) + \frac{k_r}{r}\left(\frac{H_{\nu-1}^{(1)}(k_r r) - H_{\nu+1}^{(1)}(k_r r)}{2}\right)\right\}A \\ &+ \left\{(\lambda + 2\mu)k_r^2\left(\frac{H_{\nu-2}^{(2)}(k_r r) - 2H_\nu^{(2)}(k_r r) + H_{\nu+2}^{(2)}(k_r r)}{4}\right) + \frac{k_r}{r}\left(\frac{H_{\nu-1}^{(2)}(k_r r) - H_{\nu+1}^{(2)}(k_r r)}{2}\right)\right\}B\end{aligned} \tag{5}$$

λ and μ are the Lame's constant. $e = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}$ is dilation and δ_{ij} is Kronecker delta. These displacement and stress fields provide major physical information of the wave propagation phenomena. The propagation of the fields variables otherwise can be written as

$$\begin{bmatrix} u_r \\ \sigma_{rr} \end{bmatrix} = \mathbf{M}_{Cyl}(k_r, r)\mathbf{N}_{Cyl}(k_r, r)\begin{bmatrix} A \\ B \end{bmatrix} \tag{6a}$$

where

$$\mathbf{M}_{Cyl}\left(k_r,r\right)=\begin{bmatrix} -i\omega k_r\Pi_\nu^{(1)} & -i\omega k_r\Pi_\nu^{(2)} \\ \left(\lambda+2\mu\right)k_r^2\Theta_\nu^{(1)}+\lambda\dfrac{k_r}{r}\Pi_\nu^{(1)} & \left(\lambda+2\mu\right)k_r^2\Theta_\nu^{(2)}+\lambda\dfrac{k_r}{r}\Pi_\nu^{(2)} \end{bmatrix},\quad \mathbf{N}_{Cyl}\left(k_r,r\right)=\begin{bmatrix} H_\nu^{(1)}\left(k_r r\right) & 0 \\ 0 & H_\nu^{(2)}\left(k_r r\right) \end{bmatrix} \tag{6b}$$

with

$$\Pi_\nu^{(n)}=\frac{H_{\nu-1}^{(n)}\left(k_r r\right)-H_{\nu+1}^{(n)}\left(k_r r\right)}{2H_\nu^{(n)}\left(k_r r\right)}\ \text{ and }\ \Theta_\nu^{(n)}=\frac{H_{\nu-2}^{(n)}\left(k_r r\right)-2H_\nu^{(n)}\left(k_r r\right)+H_{\nu+2}^{(n)}\left(k_r r\right)}{4H_\nu^{(n)}\left(k_r r\right)}\quad \left(n=1,2\right) \tag{6c}$$

One introduces so-called transfer matrix method for a cylindrically propagating elastic longitudinal wave. The field variable matrix at point r_1 and r_2 has linear matrix relation as

$$\begin{bmatrix} u_r \\ \sigma_{rr} \end{bmatrix}_{r_2}=\mathbf{T}_{Cyl}\left(k_r,r_1,r_2\right)\begin{bmatrix} u_r \\ \sigma_{rr} \end{bmatrix}_{r_1} \tag{7a}$$

where

$$\mathbf{T}_{Cyl}\left(k_r,r_1,r_2\right)=\mathbf{M}_{Cyl}\left(k_r,r_2\right)\mathbf{N}_{Cyl}\left(k_r,r_2\right)\mathbf{N}_{Cyl}^{-1}\left(k_r,r_1\right)\mathbf{M}_{Cyl}^{-1}\left(k_r,r_1\right) \tag{7b}$$

Here matrix $\mathbf{T}_{Cyl}$ is referred to as the transfer matrix in the cylindrical coordinate. This transfer matrix has a strong analogy to that of the Cartesian one, which can be written as

$$\mathbf{T}_{Car}\left(k_x,x_1,x_2\right)=\mathbf{M}_{Car}\left(k_x,x_2\right)\mathbf{N}_{Car}\left(k_x,x_2\right)\mathbf{N}_{Car}^{-1}\left(k_x,x_1\right)\mathbf{M}_{Car}^{-1}\left(k_x,x_1\right) \tag{8a}$$

$$\mathbf{M}_{Car}\left(k_x,x\right)=\begin{bmatrix} -\omega k_x & \omega k_x \\ -\left(\lambda+2\mu\right)k_x^2 & -\left(\lambda+2\mu\right)k_x^2 \end{bmatrix},\ \mathbf{N}_{Car}\left(k_x,x\right)=\begin{bmatrix} e^{+ik_x x} & 0 \\ 0 & e^{-ik_x x} \end{bmatrix} \tag{8b}$$

where k_x denotes wavenumber along *x*-direction and x represents a coordinate. The diagonalization of the transfer matrix $\mathbf{T}_{Car}\left(k_x,x_1,x_2\right)$ provides the Eigenvalue matrix which is equal to $\mathbf{N}_{Car}\left(k_x,r_2\right)\mathbf{N}_{Car}^{-1}\left(k_x,r_1\right)$, while one of the Eigenvector matrix is given as $\mathbf{M}_{Car}$. The Eigenvalue matrix otherwise can be calculated as

$$\mathbf{M}_{Car}\left(k_x,x\right)=-\omega k_x\begin{bmatrix} 1 & -1 \\ Z & Z \end{bmatrix} \tag{9}$$

where $Z=\rho c_p$ represents characteristic impedance of medium with density ρ and phase velocity $c_p=\sqrt{\left(\lambda+2\mu\right)/\rho}$

Here matrix $\mathbf{M}_{Cyl}\left(k_r,r_2\right)$ is assumed to be equal to one of the Eigenvector matrix of $\mathbf{T}_{Cyl}\left(k_r,r_1,r_2\right)$ while $\mathbf{N}_{Car}\left(k_x,x_2\right)\mathbf{N}_{Car}^{-1}\left(k_x,x_1\right)$ is equal to Eigenvalue matrix. This assumption is approximately valid at a far-field ($r_2,r_1\gg 1$). Similarly, the eigenvector matrix $\mathbf{M}_{Cyl}\left(k_r,r\right)$ can be simplified as

$$\mathbf{M}_{Cyl}\left(k_r,r\right)=\omega k_r\begin{bmatrix} -i\Pi_\nu^{(1)} & -i\Pi_\nu^{(2)} \\ \left(\lambda+2\mu\right)\dfrac{k_r}{\omega}\Theta_\nu^{(1)}+\lambda\dfrac{1}{r\omega}\Pi_\nu^{(1)} & \left(\lambda+2\mu\right)\dfrac{k_r}{\omega}\Theta_\nu^{(2)}+\lambda\dfrac{1}{r\omega}\Pi_\nu^{(2)} \end{bmatrix} \tag{10}$$

Since Eigenvector matrices otherwise can be calculated as

$$\bar{\mathbf{M}}_{Cyl}\left(k_r,r\right)=\omega k_r\begin{bmatrix} 1 & 1 \\ i\left(\lambda+2\mu\right)\dfrac{k_r}{\omega}\dfrac{\Theta_\nu^{(1)}}{\Pi_\nu^{(1)}}+i\dfrac{\lambda}{r\omega} & i\left(\lambda+2\mu\right)\dfrac{k_r}{\omega}\dfrac{\Theta_\nu^{(2)}}{\Pi_\nu^{(2)}}+i\dfrac{\lambda}{r\omega} \end{bmatrix} \tag{11}$$

Thus the component of $\bar{\mathbf{M}}_{Cyl}$ can be seen as an effective impedance $Z_{eff}^{(n)}$ of the cylindrical coordinate as

$$Z_{eff}^{(1)} = i\left(\lambda + 2\mu\right)\frac{k_r}{\omega}\frac{\Theta_\nu^{(1)}}{\Pi_\nu^{(1)}} + i\frac{\lambda}{r\omega}, \quad Z_{eff}^{(2)} = -i\left(\lambda + 2\mu\right)\frac{k_r}{\omega}\frac{\Theta_\nu^{(2)}}{\Pi_\nu^{(2)}} - i\frac{\lambda}{r\omega} \tag{12}$$

Similarly, the analog of the Eigen value matrices provides the calculation of the effective phase shift $\beta_{eff}^{(n)}$ as

$$\beta_{eff}^{(n)} = \angle H_\nu^{(n)}\left(k_r r_2\right) / H_\nu^{(n)}\left(k_r r_1\right) = \tan^{-1}\left(\frac{\mathrm{Im}\left(H_\nu^{(n)}\left(k_r r_2\right) / H_\nu^{(n)}\left(k_r r_1\right)\right)}{\mathrm{Re}\left(H_\nu^{(n)}\left(k_r r_2\right) / H_\nu^{(n)}\left(k_r r_1\right)\right)}\right), \left(n = 1, 2\right) \tag{13}$$

Then the effective phase velocity $c_{eff}^{(n)}$, density $\rho_{eff}^{(n)}$, and stiffness $C_{11eff}^{(n)}$ are calculated as

$$c_{eff}^{(n)} = \tan^{-1}\left(\frac{\mathrm{Im}\left(H_\nu^{(1)}\left(k_r r_2\right) / H_\nu^{(1)}\left(k_r r_1\right)\right)}{\mathrm{Re}\left(H_\nu^{(1)}\left(k_r r_2\right) / H_\nu^{(1)}\left(k_r r_1\right)\right)}\right) / \omega d, \left(n = 1, 2\right) \tag{14}$$

$$\rho_{eff}^{(n)} = 1 / c_{eff}^{(n)} Z_{eff}^{(n)}, \ \left(n = 1, 2\right) \tag{15}$$

$$C_{11eff}^{(n)} = c_{eff}^{(n)} Z_{eff}^{(n)}, \ \left(n = 1, 2\right) \tag{16}$$

Using these effective properties describe the wave propagation in a Cartesian coordinate can be seen as that in a cylindrical coordinate.

3. Results

This section is devoted to describing the effective properties of the cylindrical wave propagation and the effectiveness of the approach. Figure 1 shows the schematic drawing of the wave propagation. Axisymmetric geometry is considered to present cylindrical wave propagation. The background medium is set to be aluminum (density ρ = 2700 kg·m^{-3}, Young's modulus E =70 GPa, Poisson's ratio ν =0.33), while the medium of interest is set to be steel (ρ = 8000 kg·m^{-3}, E =200 GPa, ν =0.29). r_0 and d are the inner radius and thickness of the medium of interest, respectively. The upper and lower boundaries are subjected to a periodic condition meaning that

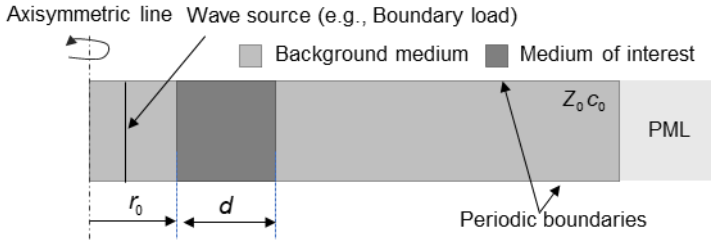

Figure 1. Schematic drawing of a medium of interest to calculate effective parameters.

Equations (12) to (16) provide the effective parameters of the target cylindrical wave. Unlike the Cartesian counterpart, the effective properties of the cylindrical wave vary regarding not only the inner radius but also the thickness of the medium and frequency. For simplicity the effect of the inner radius r_0 is focused on. Figure 2 presents the effective parameters with respect to r_0. The frequency is set to be 100 kHz and d = 5mm. The effective impedance in Figure 2(a) presents sharply varying section roughly ranging from 1×10^0 to 2×10^1. Presented by a blue shade box. It is referred to as in this section as cylindrical-dominant range. Impedance in this section has complex impedance meaning that decaying of the wave amplitude as the wave travels. As r_0 increases, the effective impedance converges to 3.96×10 kg·m^{-2}·s^{-1} and impedance imaginary converges to zero. Note that this value is the impedance in a Cartesian coordinate. This stems from the fact that wave propagation for considerably

large inner radius may be regarded as the propagation in the Cartesian coordinate. This tendency can be also found in effective phase, density, and stiffness plot in Figure 2(b) to 2(d). Their values also converge to the Cartesian one. The complex values in the cylindrical dominant section are observed.

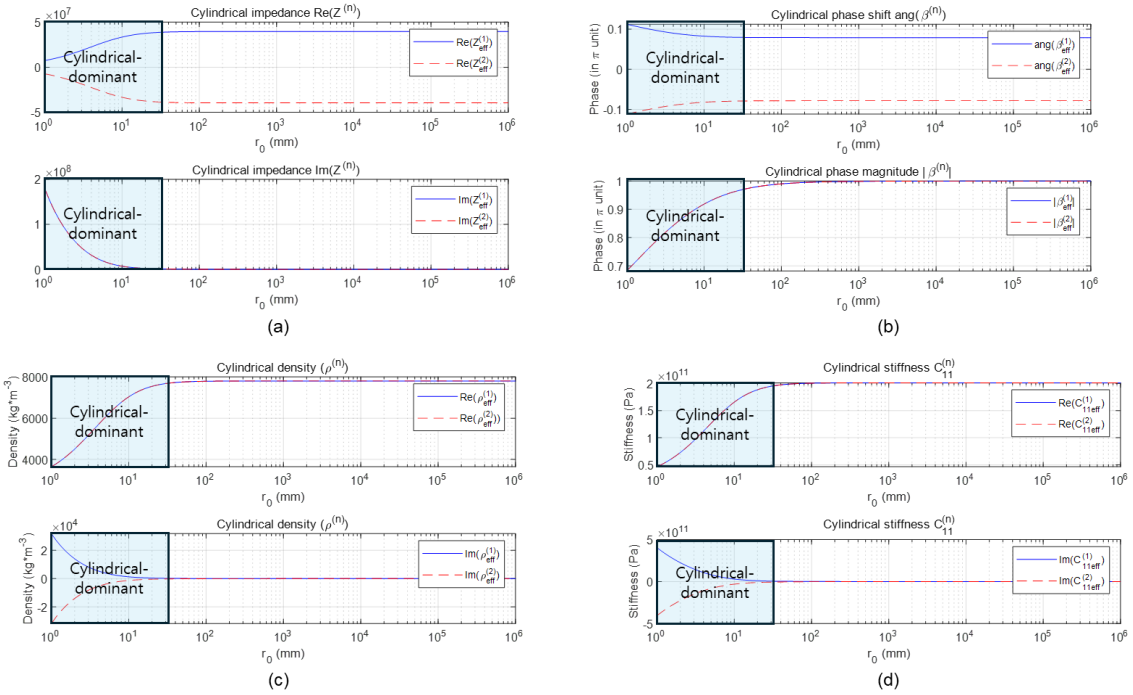

Figure 2. Effective properties that represent the wave propagation in a cylindrical coordinate: (a) Effective impedance; (b) effective phase shift; (c) effective density; (d) effective stiffness.

4. Discussion

The cylindrical effective parameters converge with Cartesian's. This convergence indicates the validity of suggested approach, since the wave propagation in considerably large size of radius can approximately be considered as Cartesian wave propagation. Much meaningful region is cylindrical-dominant range at a relatively small r_0. This varying material property indicates that translating the wave physics of a cylindrical coordinate requires complicated material properties even complex. However, considering that understanding of a near field (or small r_0) physics remains conundrum, these artificially calculated effective properties may provide clues for deeper understanding. For the future study, verification of this estimated parameters will be performed. It is considered that the comparison of the transmission and reflection coefficient over frequencies between original cylindrical wave and Cartesian estimation using effective parameters for a given r_0.

5. Conclusions

Unlike wave propagation in a Cartesian coordinate, those in a cylindrical coordinate are relatively less revealed. If one can utilize the profound understanding of Cartesian wave propagation, it provides great possibility to understand the wave physics in a cylindrical coordinate. To this end, one introduces cylindrical effective parameters—impedance, phase shift, density, and stiffness—that translate complicate wave phenomena of the cylindrical coordinate to Cartesian one based on the transfer matrix method-based approach. One utilizes the similarity of the transfer matrices in both coordinates and assumes that composition matrix of the cylindrical may be treated as equivalent one to the Cartesian one. This assumption subsequently provides cylindrical effective parameters. The obtained parameters vary considerably for the relatively small values of inner radius r_0 but converge to the Cartesian parameters for the large r_0.

Funding: This research received no external funding.

Supplementary Materials: No Supplementary Materials are presented.

Author Contributions: For research articles with several authors, a short paragraph specifying their individual contributions must be provided. The following statements should be used "Conceptualization, C.I.P.; methodology, C.I.P.; software, C.I.P.; validation, C.I.P.; formal analysis, C.I.P.; investigation, C.I.P.; resources, C.I.P.; data curation, C.I.P.; writing—original draft preparation, C.I.P.; writing—review and editing, C.I.P.; visualization, C.I.P.; supervision, C.I.P.; project administration, C.I.P. All authors have read and agreed to the published version of the manuscript.

Data Availability Statement: Not applicable.

Acknowledgments: One thanks to Y.Y. Kim and J.H. Oh for providing profound perspective on the elastic wave propagation.

Conflicts of Interest: The authors declare no conflict of interest.

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